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Polynomial-Time Semi-Rankable Sets

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Abstract

We study the polynomial-time semi-rankable sets (P-sr), the ranking analog of the P-selective sets. We prove that P-sr is a strict subset of the P-selective sets, and indeed that the two classes differ with respect to closure under complementation, closure under union with P sets, and closure under join with P sets. We also show that though P-sr falls between the P-rankable and the weakly-P-rankable sets in its inclusiveness, it equals neither of these classes.

Key words: semi-feasible sets, P-selectivity, ranking, closure properties, NNT.

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1 Introduction

In the late 1970s, Selman [Sel79] defined the semi-feasible (i.e., P-selective) sets, which are the polynomial-time analog of the Jockush's [Joc68] semi-recursive sets. Recently, there has been an intense renewal of interest in the P-selective sets and variants of the P-selective sets (see the survey [DHHT94]). Among the variants of the P-selective sets that have recently been studied are the membership comparable sets defined by Ogihara [Ogi94b], the nondeterministically selective sets [HNOSb,HHO+93], and the probabilistically selective sets defined by Wang [Wan].

However, all the variants that have been studied have been *generalizations* of the P-selective sets. This is somewhat curious as—given that the key problem with the P-selective sets is they can be quite complex—it might seem most natural to *refine* the P-selective sets and see whether the refinement retains the complexity of the P-selective sets. In this paper we do that. In particular, we look at the "polynomial-time semi-rankable sets" (P-sr), a class that is the ranking analog of the P-selective sets and is a refinement of the P-selective sets. Informally, a set A is polynomial-time semi-rankable if there is a polynomial-time two-argument function f that, whenever at least one of its inputs, say x, is in A, outputs that input and its rank within A, i.e., $||\{z \mid z \in A \text{ and } z \leq_{lexicographical} x\}||$. That is, just as a P-selective set is one that (under a certain promise condition) has certain available information regarding membership in the set, a P-sr set is one that (under the same promise condition) has certain available information regarding rank in the set.

It follows easily from the definitions that P-sr is a superset of the polynomial-time rankable sets of Goldberg and Sipser [GS91], and is a subset of the polynomial-time weakly rankable sets of Hemaspaandra and Rudich [HR90]. We prove that both these inclusions are strict. It follows immediately that all sets in P are polynomial-time semi-rankable if and only if $\mathrm{P} = \mathrm{P}^{\#\mathrm{P}}$. Further, we prove that P-sr is a *proper* refinement of the P-selective sets. Nonetheless, we also prove that the polynomial-time semi-rankable sets remain extremely complex.

Though not closed under union or join [HJ], the P-selective sets are clearly closed under union (equivalently, intersection) with P sets, under join with P sets, and under complementation. In contrast, we show that P-sr is not closed under union with P sets, under join with P sets, or under complementation. We also prove that P-sr is closed under intersection with P sets if and only if $\mathrm{P} = \mathrm{P}^{\#\mathrm{P}}$. Thus, P-sr and the P-selective sets not only differ, but even differ on very minimal natural closure properties. On the other hand, though they are a subset of the P-selective sets, we argue that P-sr seem just as hard in terms of the extended lowness hierarchy as the P-selective sets: both these classes are in the EL_2 level of the extended low hierarchy and there are oracles relative to which they are not in $\widehat{\mathrm{EL}}_2$. We also observe that the nearly near-testable sets [HH91] also lack closure under intersection (equivalently union) with P sets unless $\mathrm{P} = \mathrm{P}^{\#\mathrm{P}}$ (equivalently, $\mathrm{P} = \mathrm{NP} = \mathrm{PH} = \mathrm{P}^{\#\mathrm{P}} = \mathrm{PP}^{\mathrm{PH}}$).

2 Definitions

We let our alphabet, Σ, be $\{0,1\}$. For any set A and any string x, $A^{\leq x}$ denotes those strings in A that come before x in the standard lexicographical order. For any set A and any integer n, $A^{=n}$ denotes the strings in A of length exactly n, and $A^{\leq n}$ denotes the strings in A of length no greater than n. Let $N^{\geq 1}$ denote $\{1, 2, 3, \cdots\}$. Let TALLY $= \{A \mid A \subseteq 0^*\}$.

We state three definitions from the literature. Informally, P-selectivity captures the notion of sets for which there is a polynomial-time algorithm telling which of any two given elements is "logically no less likely to be in the set." The Goldberg-Sipser notion of polynomial-time rankability captures those sets that are sufficiently simple that there is a polynomial-time algorithm that can determine (for elements in the set) the number of elements in the set up to that point. There have been many papers studying the issue of which sets can be ranked [GS91,HR90,BGS91,Huy90].

Definition 2.1

1. **[GS91]** For any set B and any string x, define $rank_B(x) = ||B^{\leq x}||$. A set A is *P-rankable* if there is a polynomial-time computable function f such that (a) $(\forall x \in A)\,[f(x) = rank_A(x)]$ and (b) $(\forall x \notin A)\,[f(x) =$ "not in A"$]$. We also use P-rankable to denote the class of sets that are P-rankable.

2. **[HR90]** A set A is *weakly-P-rankable* if there is a polynomial-time computable function f such that $(\forall x \in A)\,[f(x) = rank_A(x)]$. We also use weakly-P-rankable to denote the class of sets that are weakly-P-rankable.

Note that for $x \notin A$, the definition of weakly-P-rankable sets puts no constraint on the behavior of f on x other than that it must run in polynomial time.

Definition 2.2 [Sel79,Sel82] A set A is *P-selective* if there is a (total, single-valued) polynomial-time computable function f such that, for every x and y, it holds that

1. $f(x,y) = x$ or $f(x,y) = y$, and
2. $\{x,y\} \bigcap A \neq \emptyset \Rightarrow [(x \in A \text{ and } f(x,y) = x) \text{ or } (y \in A \text{ and } f(x,y) = y)]$.

We also use P-selective to denote the class of sets that are P-selective.

The above definition is more verbose than needed, so as to bring out the analogy with the P-sr sets.

We define the following refinement of P-selectivity. This refinement requires the production not just of a member of the set (under a certain hypothesis), but also (under the same hypothesis) the accompanying rank information giving the location within the set of the member.

Definition 2.3 A set A is *polynomial-time semi-rankable* if there is a (total, single-valued) function f such that, for every x and y,

1. $(\exists n)\,[f(x,y) = \langle x,n\rangle \text{ or } f(x,y) = \langle y,n\rangle]$, and

2. $\{x, y\} \bigcap A \neq \emptyset \Rightarrow [(x \in A \text{ and } f(x, y) = \langle x, rank_A(x)\rangle) \text{ or } (y \in A \text{ and } f(x, y) = \langle y, rank_A(y)\rangle)]$.

In such a case, we say that f is a semi-ranking function for A. We use P-sr to denote the class of sets that are polynomial-time semi-rankable.

The following result is immediate.

Proposition 2.4 P-sr = P-selective $\cap$ weakly-P-rankable.

Though we adopt Definition 2.3 throughout this paper, we note that the definition is relatively robust. For example, if one deletes the definition's condition 1 the class of languages defined remains unchanged, and if one deletes condition 1 and changes the hypothesis of condition 2 to "$\{x, y\} \bigcap A \neq \emptyset$ and $x \neq y$" the class of languages defined also remains unchanged.

It follows immediately from the definitions that P-rankable $\subseteq$ P-sr $\subseteq$ weakly-P-rankable and P-sr $\subseteq$ P-selective. From this and the result that all P sets are P-rankable if and only if all P sets are weakly-P-rankable if and only if $\mathrm{P} = \mathrm{P}^{\#\mathrm{P}}$ ([HR90], see also [GS91]), we have the following.

Proposition 2.5 All P sets are polynomial-time semi-rankable if and only if $\mathrm{P} = \mathrm{P}^{\#\mathrm{P}}$.

Ko proved that all P-selective sets have small circuits (i.e., P-selective $\subseteq$ P/poly). It is not hard to see that all P-sr sets have small ranking circuits (i.e., P-sr $\subseteq$ P/poly-rankable, where the P/poly is in fact representing the function class FP/poly in the same way the P in P-selective represents the function class FP).

Note that if f' is a semi-ranking function for A, then $f(x, y) =_{def} f'(min(x, y), max(x, y))$ is a semi-ranking function for A having the property that for every x and y, $f(x, y) = f(y, x)$. We assume that all semi-ranking functions discussed henceforward are already in this "oblivious to the ordering of their arguments" form.

We review the definitions of the low and extended low hierarchies to which we will refer in the last part of the paper. Following Ko and Schöning [KS85], for all $k \geq 0$ we define L_k to be the class of sets L in NP such that $\Sigma_k^{p,L} = \Sigma_k^p$, and $\widehat{\mathrm{L}}_k$ is the class of sets L such that $\Delta_k^{p,L} = \Delta_k^p$. Thus the sets in the low hierarchy are those sets in NP that provide no additional power to some level of the polynomial hierarchy, when given as an oracle. To help classify sets that are not in NP, the extended low hierarchy was defined by Balcázar, Book, and Schöning [BBS86] as follows: For all $k \geq 1$, EL_k is the class of sets L such that $\Sigma_k^{p,L} \subseteq \Sigma_{k-1}^{p,L\oplus\mathrm{SAT}}$, where $A \oplus B = \{0x \mid x \in A\} \cup \{1x \mid x \in B\}$. Similarly, one can define intermediate levels, as suggested by Schöning in [Sch86]. Let $\widehat{\mathrm{EL}}_k$ denote the class of sets such that $\Delta_k^{p,L} \subseteq \Delta_{k-1}^{p,L\oplus\mathrm{SAT}}$. The relativized versions with respect to oracle A of EL_k and $\widehat{\mathrm{EL}}_k$ are obtained by replacing SAT by some standard complete set for NP^A.

3 Separations

The polynomial-time semi-rankable sets are a proper refinement of the P-selective sets, as shown by the following result.

Theorem 3.1 P-sr $\subsetneq$ P-selective.

Proof: Note that P-sr $\subseteq$ P-selective, since we can obtain a P-selector function from a P-sr function by simply ignoring the rank information. We will show that there exists a set that is P-selective but not P-sr. Define $\mu(1) = 2$, and $\mu(i+1) = 2^{2^{\mu(i)}}$ for each $i \geq 1$. Let $\{f_i\}_{i \in N^{\geq 1}}$ be a standard enumeration of all polynomial-time 2-ary transducers, and let this enumeration have the property that each transducer is repeated infinitely often. Let $sA = \{sx | x \in A\}$, and let the join (sometimes referred to in the literature as disjoint union or marked union) operator be defined by $A \oplus B = 0A \cup 1B$. We will construct, in stages, a set $A = \bigcup_{i \geq 0} A_i$, and we will argue that $A \oplus 1^* \in$ P-selective$-$P-sr. We will construct A so that it satisfies the following conditions:

1. $A \in \mathrm{E}$, where $\mathrm{E} = \bigcup_{c \geq 0} \mathrm{DTIME}[2^{cn}]$, and
2. $A \subseteq H$, where $H = \{0^{\mu(1)}, 0^{\mu(2)}, 0^{\mu(3)}, \ldots\}$.

STAGE 0: Let $A_0 = \emptyset$.
STAGE i, $i \in N^{\geq 1}$**:** Run $f_i(1^{\mu(i)+1}, 1^{\mu(i)+1})$ for at most $2^{\sqrt[3]{(\mu(i)+1)}}$ steps. (The root is to ensure that the small overhead of simulating a machine causes us no problems.) If it has not finished within this time, then set $A_i = A_{i-1}$ and go to the next stage. If it finishes running within this time, then let $\langle w, n \rangle$ denote its output. If $w \neq 1^{\mu(i)+1}$, then f_i is not a P-sr function for $A_{i-1} \oplus 1^*$, since clearly $1^{\mu(i)+1} \in A_{i-1} \oplus 1^*$; set $A_i = A_{i-1}$ and go to the next stage. If $w = 1^{\mu(i)+1}$, then let $q = rank_{A_{i-1} \oplus 1^*}(w)$. Notice that there are exactly $a_1 = \mu(i)$ strings in $(A_{i-1} \oplus 1^*) \cap 1\Sigma^*$ that are lexicographically smaller than w, and by brute force we can compute $a_2 = ||A_{i-1}{}^{\leq \mu(i-1)}||$, which is the number of strings in $(A_{i-1} \oplus 1^*) \cap 0\Sigma^*$ that are lexicographically smaller than w. Thus, $q = a_1 + a_2 + 1$ is computable in time polynomial in $|w|$. Now, if $n \neq q$, then clearly f_i is not a P-sr function for $A_{i-1} \oplus 1^*$. Let $A_i = A_{i-1}$ and go to the next stage. Otherwise, $n = q$. Let $A_i = A_{i-1} \cup \{0^{\mu(i)}\}$. By our construction, the rank of w will now be $q + 1$, which makes the output of f_i wrong; go to the next stage.

Note that the time cutoff for f_i in stage i ensures that $A \in E$, and since each transducer is repeated infinitely often in the enumeration, running out of time is not a problem, as for all but a finite number of occurrences of each transducer we will not run out of time. By our construction above, $A \oplus 1^* \notin$ P-sr, since each potential P-sr transducer is eventually eliminated (and the diagonalizations against $A_i \oplus 1^*$ hold against $A \oplus 1^*$ by construction).

However, $A \oplus 1^* \in$ P-selective via the following P-selector function:

$$h(x,y) = \begin{cases} x & \text{if } x, y \notin H \oplus 1^* \\ x & \text{if } x \in 11^* \\ y & \text{if } x \notin 11^* \text{ and } y \in 11^* \\ x & \text{if } x = y \in 0H \\ min(x,y) & \text{if } x, y \in 0H, x \neq y, min(x,y) \in 0A \\ max(x,y) & \text{if } x, y \in 0H, x \neq y, min(x,y) \notin 0A \end{cases}$$

Note that since, if $x \neq y$ and $x, y \in 0H$, $max(|x|,|y|) \geq 2^{2^{min(|x|,|y|)}}$, we can in this case decide by brute force whether $min(x,y) \in 0A$. Thus, $h(x,y)$ is computable in time polynomial in $max(|x|,|y|)$. ∎

Though Theorem 3.1 shows that P-sr differs from the class of P-selective sets, one can well ask if they differ in natural ways. Later, we will show that they differ even with respect to some quite minimal closure properties.

The fact (Theorem 3.1) that the polynomial-time semi-rankable sets properly refine the P-selective sets notwithstanding, P-sr contains quite complex sets.

Theorem 3.2 Let f be any (total) recursive function. Then P-sr $\not\subseteq$ DTIME$[\mathcal{O}(f(n))]$.

Proof: We will show that there exists a set B, such that $B \in$ P-sr, but $B \notin$ DTIME$[\mathcal{O}(f(n))]$. It is well-known that for any given recursive function f, it holds that

$(\exists g)\,(\forall \widehat{h} = \mathcal{O}(f))\,(\exists n_0 \in N^{\geq 1})$:

1. g is strictly monotonically increasing,
2. g is a (total) recursive function, and
3. $(\forall n \geq n_0)[\widehat{h}(n) < g(n)]$.

In particular, let M be a machine computing recursive function f. We may define $g(0) = 2^{max(1, \mathrm{runtime}_M(0))}$ and, inductively, for $i \geq 0$, $g(i+1) = 2^{(i+1)\,max(g(0), \cdots, g(i), \mathrm{runtime}_M(i+1))}$. Note that this g has the property that $\{i\#0^j \mid g(i) \leq j\} \in$ P. Define $\mu(1) = 2$ and, for $i \geq 1$, define inductively $\mu(i+1) = g(\mu(i))$. Let $H = \{0^{\mu(1)}, 0^{\mu(2)}, 0^{\mu(3)}, \ldots\}$. Note that $H \in$ P. Our construction will ensure that $B \subseteq H$.

Let $\alpha(s) = ||\{z | z \in B \text{ and } z <_{lexicographical} s\}||$, i.e., $\alpha(s)$ is the number of elements of B that are lexicographically strictly less than s.

Let $\{\widehat{M_i}\}_{i \in N^{\geq 1}}$ be a standard enumeration of all deterministic Turing machines. As before, we desire every machine to appear infinitely often in our enumeration; so define a new enumeration $\{M_i\}_{i \in N^{\geq 1}}$ by $M_{\langle j,k \rangle} = \widehat{M_j}$, where $\langle \cdot, \cdot \rangle$ is any easily computable and easily invertible bijection between $N^{\geq 1} \times N^{\geq 1}$ and $N^{\geq 1}$. We construct $B = \bigcup_{i \geq 0} B_i$, in stages, such that $(\forall \widehat{h}(n) = \mathcal{O}(f(n)))[B \notin \mathrm{DTIME}[\widehat{h}(n)]]$.

STAGE 0: Let $B_0 = \emptyset$.

STAGE i, $i \in N^{\geq 1}$: Run M_i on input $x = 0^{\mu(i)}$. If M_i accepts it within $g(|x|)$ steps, then let $B_i = B_{i-1}$, else let $B_i = B_{i-1} \cup \{0^{\mu(i)}\}$.

$B \in$ P-sr, via the semi-ranking function:

$$\ell(x,y) = \begin{cases} \langle x, 1\rangle \text{ or } \langle y, 1\rangle & \text{if } \{x,y\} \cap H = \emptyset \\ \langle x, 1+\alpha(x)\rangle & \text{if } x \in H, y \notin H \\ \langle y, 1+\alpha(y)\rangle & \text{if } y \in H, x \notin H \\ \langle x, 1+\alpha(x)\rangle & \text{if } x = y \in H \\ \langle min(x,y), 1+\alpha(min(x,y))\rangle & \text{if } x,y \in H, x \neq y, min(x,y) \in B \\ \langle max(x,y), 1+\alpha(max(x,y))\rangle & \text{if } x,y \in H, x \neq y, min(x,y) \notin B \end{cases}$$

Note that for all $x, y \in H$, if $x < y$ then $|y| \geq g(|x|)$. So for each $h = \mathcal{O}(f(n))$ and for each machine $M_{\langle j,k \rangle}$ in our enumeration such that $M_{\langle j,k \rangle}$ has runtime bounded by h, for all but a finite number of $M_{\langle j,1 \rangle}$, $M_{\langle j,2 \rangle}$, $M_{\langle j,3 \rangle}$, $\cdots$ we diagonalize successfully (and thus implicitly diagonalize against $M_{\langle j,k \rangle}$). Note that ℓ is computable in time polynomial in $max(|x|, |y|)$, and that $\ell(x, y)$ is also a P-sr function for B. ∎

Note that the B of the proof of Theorem 3.2 was a tally set. Thus, in the statement of Theorem 3.2 one can make the stronger claim P-sr $\cap$ TALLY $\not\subseteq$ DTIME$[\mathcal{O}(f(n))]$.

Theorem 3.2 gives one type of P-sr set that can be kept out of P. Another example, somewhat analogous to the role left cuts play for the P-selective sets, would be "widely spaced and easy" left cuts. By this we mean sets containing only elements at appropriately widely spaced lengths (as in the proof of Theorem 3.2), and with the set at each of these lengths being the left cut (at that length) of a real number (the same at each length), and with the complexity of the number being such that at each nonempty length, one can brute-force compute the cut point at the previous nonempty length.

If P-sr $\subseteq$ P-rankable then P-sr $\subseteq$ P, as all P-rankable sets are in P. But this would contradict Theorem 3.2. So, since P-rankable $\subseteq$ P-sr as already observed, we have the following corollary.

Corollary 3.3 P-rankable $\subsetneq$ P-sr.

Similarly, the inclusion P-sr $\subseteq$ weakly-P-rankable is also strict.

Theorem 3.4 P-sr $\subsetneq$ weakly-P-rankable.

Proof: Note that P-sr $\subseteq$ weakly-P-rankable, since we can construct a weakly-P-rankable function from a P-sr function f for a given set by returning the rank output by $f(x, x)$. We will show that there exists a set B such that $B \in$ weakly-P-rankable, but $B \notin$ P-sr. Consider any set B such that $(\forall n \geq 1)[||B^{=n}|| = 1]$. Then $B \in$ weakly-P-rankable via the function, $(\forall x)\,[h(x) = |x|]$, since if $x \in B$, then $rank_B(x) = |x|$.

Let $\{f_i\}_{i \in N^{\geq 1}}$ be a standard enumeration of all polynomial-time 2-ary functions. We will now construct, in stages, a particular set $B = \bigcup_{i \geq 0} B_i$, satisfying the above property:
STAGE 0: $B_0 = \emptyset$.
STAGE i, $i \in N^{\geq 1}$: Suppose $f_i(0^{2i-1}, 0^{2i}) = \langle w, n \rangle$. If $w = 0^{2i-1}$, then let $B_i = B_{i-1} \cup \{1^{2i-1}, 0^{2i}\}$, making the output of f_i wrong, since $w \notin B_i$. If $w = 0^{2i}$, then let $B_i = B_{i-1} \cup \{0^{2i-1}, 1^{2i}\}$, making the output of f_i wrong, since $w \notin B_i$. Otherwise, i.e., if $w \neq 0^{2i-1}$ and $w \neq 0^{2i}$, let $B_i = B_{i-1} \cup \{0^{2i-1}, 0^{2i}\}$; the output of f_i is clearly wrong in this case.

Since at each stage i, $i > 0$, we add to B exactly one string at length $2i - 1$ and $2i$, B has the desired one-per-length property, and clearly $B \notin$ P-sr, as each potential P-sr function fails at some stage. ∎

4 Closure Properties

Theorem 3.1 shows that the P-sr sets and the P-selective sets are different classes. Yet, one may wonder whether they differ on natural properties. In fact, they differ sharply regarding closure properties. Though Hemaspaandra and Jiang [HJ] have noted that the P-selective sets are not closed under union (equivalently, due to closure under complementation, intersection) or join, the P-selective sets clearly are closed under complementation, and under union (equivalently, intersection) with P sets. In contrast, P-sr is not closed under union with P sets, under join with P sets, or under complementation.

Theorem 4.1 P-sr is not closed under union with P sets, under join with P sets, or under complementation.

Proof: Let B and H be the sets B and H from the proof of Theorem 3.2 for the case where the f of that theorem is some time-constructible function that majorizes all polynomials, e.g., $f(n) = 2^n$. Recall that $B \in$ P-sr and that $H \in$ P. Recall that $sA =_{\mathrm{def}} \{sx | x \in A\}$, and that the join operation is defined as $F \oplus G =_{\mathrm{def}} 0F \cup 1G$. Suppose $B \oplus H$ is in P-sr. Let $k(\cdot)$ denote some polynomial-time semi-ranking function for $B \oplus H$. Then to determine in polynomial time whether an arbitrary string x is in B, we can do the following. If $x \notin H$ then $x \notin B$. If $x \in H$, run $k(0x,\ 10^{|x|})$. If the output is $0x$ along with a rank, then $x \in B$. If the output is $10^{|x|}$ along with a rank, then due to the construction of B it is easy to determine via brute force exactly how many strings are in $B \oplus H$ up to $10^{|x|}$ excluding $0x$. Thus, x is in B exactly if this number is one less than the rank k returned. It is not too hard to see (considering the strong relationship between the properties of B and the properties of $0B$) that the above also establishes that P-sr is not closed under union with P sets. Similarly, if the complement of B were in P-sr, B clearly is in P, via using the semi-ranker for $\overline{B}$ on the two strings lexicographically following any given element of H in whose membership in B one is interested. ∎

It is somewhat surprising that deciding the closure of P-sr under intersection with P sets is a much more difficult problem.

Theorem 4.2 P-sr is closed under intersection with P sets if and only if $\mathrm{P} = \mathrm{P}^{\#\mathrm{P}}$.

Proof: If $\mathrm{P} \neq \mathrm{P}^{\#\mathrm{P}}$, then by Proposition 2.5, there is a set B in P which is not polynomial-time semi-rankable. Then Σ^* is in P-sr but $\Sigma^* \cap B$ is not.

Suppose now that $\mathrm{P} = \mathrm{P}^{\#\mathrm{P}}$. So P = NP = coNP. Let A be a set in P-sr via the function f and B a set in P. Clearly, $A \cap B$ is P-selective. By Proposition 2.4, we have only to show that $A \cap B$ can be weakly ranked in polynomial time. Let $r(x)$ be defined by $f(x,x) = \langle x, r(x)\rangle$ and $s(x,y)$ be defined by $f(x,y) = \langle s(x,y), n\rangle$ for some natural n (i.e., we have taken the ranking and the selector functions of A separately). Let $C = \{(x,y) \in \Sigma^* \times \Sigma^* \mid y \in B \text{ and } y \leq x \text{ and } r(y) \leq r(x) \text{ and } (\forall z \leq x)\, [r(z) = r(y) \Rightarrow s(z,y) = y]\}$. Observe that C is a coNP set and thus, by our assumption, is in P. Let $g(x) = ||\{y \mid (x,y) \in C\}||$. Clearly, g is computable by a #P computation with access to C and so, again by our assumption, g is computable in polynomial time, as if $\mathrm{P} = \mathrm{P}^{\#\mathrm{P}}$ then $\mathrm{FP} = \mathrm{FP}^{\#\mathrm{P}}$. Now, observe that if $x \in A \cap B$ then $g(x) = rank_{A \cap B}(x)$. This holds as if $x \in A \cap B$ then $\{y \mid (x, y) \in C\} = \{y \mid y \leq x \text{ and } y \in A \cap B\}$. ∎

Theorem 4.1 and Theorem 4.2 show that P-sr lacks even certain very minimal closure properties. Do other already-defined classes also lack such minimal closure properties, or is P-sr unique in this regard? In this regard, we make the following two observations. The first one contrasts interestingly with Theorem 4.2 in light of the fact that P-sr = P-selective $\cap$ weakly-P-rankable.

Observation 4.3 The class weakly-P-rankable is not closed under intersection with P sets.

Proof: Build A in stages.

STAGE i, $i \in N^{\geq 1}$**:** Let $m_{i-1} = ||A_{i-1} \cap (\Sigma^* - 0^*)||$ and let f_i be the ith polynomial-time transducer. If $f_i(110^i) \neq m_{i-1} + 1$, then add 000^i and 110^i to A. If $f_i(110^i) = m_{i-1} + 1$, then add 010^i and 110^i to A. Then A is weakly rankable, but $A \cap (\Sigma^* - 0^*)$ is not. ∎

We claim that NNT, the class of sets having polynomial-time "implicit membership tests," also lacks such minimal closure properties under reasonable complexity-theoretic assumptions. NNT [HH91] is the class of all sets A such that A has a polynomial-time computable function f that on each input x states (correctly) either that $x \in A$, or that $x \notin A$, or that exactly one of x and the lexicographical predecessor of x is in A, or that not exactly one of x and the lexicographical predecessor of x is in A.

Observation 4.4 $\mathrm{P} = \mathrm{NP} = \mathrm{PH} = \mathrm{P}^{\#\mathrm{P}}$ if and only if NNT is closed under intersection (equivalently, union) with P sets if and only if NNT is closed under join with P sets.

Proof: First, since NNT is in $\mathrm{P}^{\#\mathrm{P}}$, the fact that $\mathrm{P} = \mathrm{P}^{\#\mathrm{P}}$ implies the other two conditions is immediate. If NNT is closed under intersection with P sets—indeed, under intersection with the very simple set $(0+1)^*0$—then clearly P = NNT. By combining two results of [HH91] (namely, the characterization of $\oplus$OptP—which is now known (see the discussion in [HO94]) to be equivalent to the serializability class [CF91,Ogi94a] SF_2—in terms of NNT, and the observation regarding the consequences of NNT = $\oplus$OptP) it follows that $\mathrm{P} = \mathrm{PP}^{\mathrm{PH}}$. The same argument holds for closure under disjoint union with P sets—indeed with the trivial set $\emptyset$. ∎

Although the classes P-sr and P-selective differ with respect to some simple operations, their lowness properties are similar. Ko and Schöning [KS85] proved that all sets in P-selective $\cap$ NP are in the L_2 level of the low hierarchy, and Amir, Beigel, and Gasarch [ABG90] proved that all sets in P-selective are in the EL_2 level of the extended low hierarchy. Allender and Hemaspaandra [AH92] have built oracles relative to which P-selective $\cap$ NP is not in $\widehat{\mathrm{L}}_2$ and P-selective is not in $\widehat{\mathrm{EL}}_2$. In the absence of oracles, such a result is currently beyond reach, because it was shown by E. Hemaspaandra, Naik, Ogihara, and Selman [HNOSa] that if P = PP, then every P-selective set is $\leq_T^P$ equivalent to a tally set and thus is in $\widehat{\mathrm{EL}}_2$ [BB86]. We show that P-sr has the same properties as P-selective with respect to the extended low hierarchy: clearly, P-sr is in EL_2 and P-sr $\cap$ NP is in L_2 (because polynomial-time semi-rankable sets are P-selective) and as we show below there is an oracle relative to which P-sr is not in $\widehat{\mathrm{EL}}_2$. The problem of finding a similar relativized lower bound on the location of P-sr in the low hierarchy is open.

Theorem 4.5 There is an oracle A relative to which P-sr is not in $\widehat{\mathrm{EL}}_2$.

Proof: Let $\{N_i\}_{i\in N^{\geq 1}}$ be an enumeration of all polynomial-time oracle nondeterministic machines such that for any oracle A, for all i, and for all n the machine N_i^A runs for at most $n^i + i$ steps on all inputs of length n. Then for each oracle A, the set $K(A) = \{\langle i, x, 1^{|x|^i+i}\rangle \mid N_i^A \text{ accepts } x\}$ is NP^A-complete. We build an oracle A such that the following two statements are fulfilled:

(1) $L(A) = \{x \mid (\forall y)\,[|y| = |x| \text{ and } 0xy \in A]\}$ is P^A-sr,

(2) $B(A) = \{0^n \mid (\exists x \in \Sigma^n)\,[\, x \in L]\}$ is not in $\mathrm{P}^{K(A)}$.

Since $B \in \mathrm{NP}^{L(A)} \subseteq \mathrm{P}^{\mathrm{NP}^{L(A)\oplus A}}$ and $B \notin \mathrm{P}^{K(A)} = \mathrm{P}^{L(A)\oplus K(A)\oplus A}$ (the last equality follows from $L(A) \in \mathrm{P}^{K(A)}$ and $A \in \mathrm{P}^{K(A)}$), we have that $L(A)$ is not in $\widehat{\mathrm{EL}}_2^{\,A}$.

Statement (1) will be met in the following way. Let $\mu(i)$ be the sequence defined by $\mu(0) = 1$ and $\mu(i+1) = 2^{\mu(i)}$ for $i \geq 0$, and let $J = \{\mu(i) \mid i \geq 0\}$. The oracle A is constructed in such a way as to guarantee that: (i) if $x \in L(A)$ then $|x| \in J$, (ii) for each $\mu(i) \in J$, there is at most one string of length $\mu(i)$ in $L(A)$, and (iii) if $x \in L(A)$ then for all strings y with $|y| = |x|$ and $y \neq x$, $1\langle x, y\rangle \in A$ and $1\langle y, x\rangle \notin A$. Since, clearly, $L(A)$ belongs to $\mathrm{DTIME}^A[2^n]$, standard arguments show that $L(A)$ is P^A-sr. A is constructed in stages. At each moment in the construction, we consider only those extensions of the oracle built so far that preserve the above conditions (i), (ii), and (iii) for the initial segment of L that has been (implicitely) built up to that moment. Such extensions are called *legal* extensions. We denote by n_j the length up to which the membership of strings in A has been established by the end of stage j. Let $\{P_j\}_{j\in N^{\geq 1}}$ be an enumeration of all polynomial-time oracle deterministic machines such that for all oracles O, for all j, and for all n, the machine N_i^O runs for at most $n^j + j$ steps on all inputs of length n.

STAGE 0: $A = \emptyset$, $n_0 = 0$.

STAGE j, $j \in N^{\geq 1}$**:** Choose $n \in J$ sufficiently large so that $n > n_{j-1}$ and $(n^j + j)^2 < 2^n$. Reserve all strings having length between $n_{j-1}+1$ and $n-1$ for $\overline{A}$, the complement of A. Note that this is a legal extension.

Next, $P_j^{K(A)}$ is simulated on input 0^n. Let w_1 be the first query to the oracle set. If w_1 is not of the form $\langle i, x, 1^{|x|^i+i}\rangle$, then answer NO and continue the simulation. Suppose that $w_1 = \langle i, x, 1^{|x|^i+i}\rangle$ for some x and i. Observe that $|x|^i + i \leq n^j + j$. If there is a legal extension S of A such that N_i^S accepts x, then choose one accepting path of N_i on x with oracle S and let Q be the set of strings queried along this path. Reserve all strings in $Q \cap S$ for A , and reserve for $\overline{A}$ all strings in $Q \cap \overline{S}$. At most $|x|^i + i \leq n^j + j$ strings are reserved in this way for either A or $\overline{A}$. Now, $w_1 \in K(A)$ and the simulation can be continued with the YES answer. If there is no such legal extension S of A do not reserve any strings for A or $\overline{A}$, answer NO to the query and continue the simulation. Note that whatever legal extension of A will be taken in the future, the answer NO remains correct. Proceed in the same way with all queries in the simulation. Since there are at most $n^j + j$ queries and each query reserves at most $n^j + j$ strings for A or $\overline{A}$, the whole simulation reserves at this stage less than $(n^j + j)^2 < 2^n$ strings for A or $\overline{A}$. Note that if for some pair x, y with $|x| = |y|$, $0xy$ is reserved for $\overline{A}$, or $1\langle x, y\rangle$ is reserved for $\overline{A}$ and $x \neq y$, or $1\langle y, x\rangle$ is reserved for A and $x \neq y$, then x is forced to belong to $\overline{L(A)}$. A string x of length n could be forced to belong to $L(A)$ only if $0xy$ is reserved for A for all y of length n and this is not possible because at most $(n_j + j)^2 < 2^n$ strings are reserved for A. Consequently, no string x is forced to belong to $L(A)$ and at most $(n^j + j)^2$ strings may be forced to belong to $\overline{L(A)}$. There are two cases to analyze next.

Case 1. The simulation of $P_j^{K(A)}$ accepts 0^n. Since no string x is forced to belong to $L(A)$, there is a legal extension of A such that $L(A)$ contains no string of length n. Take such an extension that reserves to A or $\overline{A}$ all strings of length less than or equal to $(n^j + j)^2$, let $n_j = (n^j + j)^2$, and go to the next stage. Since $O^n \notin B(A)$, it is guaranteed that $B(A) \neq L(P_j^{K(A)})$.

Case 2. The simulation of $P_j^{K(A)}$ rejects 0^n. Since less than 2^n strings of length n are forced to belong to $\overline{L(A)}$ by the simulation, there exists an x of length n that is not forced to be in $\overline{L(A)}$. Extend A legally so that $x \in L(A)$ and the membership in A of all strings of length less than or equal to $(n^j + j)^2$ is decided by this extension, take $n_j = (n^j + j)^2$, and go to the next stage. Now, $0^n \in B(A)$ and, thus, again, $B(A) \neq L(P_j^K(A))$.

Clearly, this construction satisfies statement (2). ∎

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